rupture occurs to the left of (1.19) only when the curves (2.11) and (2.17) drop and shrink to the "triple" point $\gamma = 2\gamma_*, M = \operatorname{tg} 2\gamma_*$. The line (2.9) which will shift to the left as σ_* decreases will be the boundary separating the solution with rupture from the solution without rupture.

Thus, for very large values of σ_* rupture is possible only in the second and third regimes for high impact velocities, the curve of the limit states is in the domain of large values of M. As σ_* decreases, this curve drops montonically, and for a certain σ_* shrinks into a triple point. As σ_* decreases further, it is transformed into the segment of a line (2.9) which tends to the axis M as $\sigma_* \to 0$.

The solution of corresponding problems on the impact of a cone on a membrane can be constructed by exactly analogous methods by using the singularities of the solution at the break point of the structure and the scheme taken for the rupture process.

The results obtained here can be utilized in the general case of non-selfsimilar problems with curvilinear outlines of the impacting body surface.

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ON THE THEORY OF LONG WAVES IN AN INCLINED CHANNEL*

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A method resembling the asymptotic small-parameter method /1—3/, is used to study the long steady waves in an inclined channel, with the waves degenerating into solitons as their length tends to infinity. By analogy with the theory of stability of elastic rods, the process of transition from one-dimensional steady flow to two-dimensional flow, can be represented as instantaneous, with the result that all rectilinear stream lines becomes curved, but the values of the Froude and Reynolds numbers remain the same. It is shown that solutions of this type can exist, provided that the velocity of wave propagation and the value of the Reynolds number are nearly critical. Simple formulas are obtained for the wave profile, and the dependence of the wave propagation on the amplitude. If the Reynolds number is small and the angle of inclination of the channel is nearly $\pi/2$, the same formulas hold even without the assumption that the Reynolds number is nearly critical. The method opens up the possibility of proving existence and uniqueness theorems by analogy with /1—3/. Technical difficulties arise in connection with the estimates for Green's function for the biharmonic operator.

1. Formulation of the problem. Consider the two-dimensional steady flow of a homogeneous, incompressible heavy viscous fluid with a free boundary, over a rectilinear bottom inclined at an angle α to the horizontal. We shall assume that the two-dimensional flow is caused by instantaneous loss of stability of a one-dimensional flow characterized by the Reynolds number $R = Q/\nu$ and Froude number $F = gH^3/Q^2$ (Q is the flow rate and H is the depth of the stream. We shall write the equations of motion in a coordinate system moving in a direction parallel to the channel bottom with wave velocity c. The origin of coordinates is chosen at the free unperturbed boundary, and the y axis is parallel to the force of gravity.

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The rate of flow is determined in the fixed coordinate system.

We have the following formulas for the velocity of one-dimensional flow /4/:

$$a(\eta)=\frac{3}{2}(\eta^2-1)+c=\frac{3}{2}(\eta^2+\lambda),\quad \lambda=\frac{2}{3}c-1 \tag{1.1}$$

(η is the ordinate). We also introduce the following stream function for one-dimensional flow:

$$\frac{d\psi}{d\eta}=a(\eta)=\frac{3}{2}\eta^2+\lambda,\quad \psi(0)=0,\quad \psi=\frac{1}{3}\eta^3+\frac{3}{2}\lambda\eta$$

If $\lambda>0$, then $a(\eta)>0$ and the following inverse function exists:

$$\eta=\eta(\psi),\quad 0\leqslant\psi\leqslant\psi_0=\frac{1}{2}+\frac{3}{2}\lambda,\quad \frac{d\eta}{d\psi}=1/a(\eta) \tag{1.2}$$

2. Transformation of the equations of two-dimensional flow. Introducing the stream function $\psi(x, y)$ for the two-dimensional flow and using the relation $FR\sin\alpha=3$ [4], we can write the Navier-Stokes equations in the form

$$\begin{gathered}\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=-\frac{3}{R}+\frac{1}{R}\frac{\partial(\Delta\psi)}{\partial y}-\frac{\partial p}{\partial x}\\ -\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2}+\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\,\partial y}=\frac{3\operatorname{ctg}\alpha}{R}-\frac{1}{R}\frac{\partial(\Delta\psi)}{\partial x}-\frac{\partial p}{\partial y}\end{gathered} \tag{2.1}$$

The tangential stresses at the free boundary $y=Y(x)$ must be equal to zero. We shall write this condition in the form

$$\begin{gathered}y=Y(x),\quad p=\frac{2}{R}\frac{1+Y'(x)^2}{1-Y'(x)^2}\frac{\partial^2\psi}{\partial x\,\partial y}\\ \frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}-\frac{4Y'(x)}{1-Y'(x)^2}\frac{\partial^2\psi}{\partial x\,\partial y}=0\end{gathered} \tag{2.2}$$

The free boundary and the bottom must represent the stream lines, and the condition of adhesion must hold at the bottom

$$\psi(x, Y(x))=0,\quad \psi(x,1)=\frac{1}{2}+\frac{3}{2}\lambda,\quad \frac{\partial\psi}{\partial y}(x,1)=c \tag{2.3}$$

The difference encountered in the course of solving the non-linear boundary value problem (2.1)–(2.3) are caused by the fact that the free boundary $y=Y(x)$ is not known and must be determined during the solution. The problem becomes more complicated in the non-steady and the three-dimensional case, and also when the surface tension at the free boundary is taken into account.

We note that the formulation of the problem (2.1)–(2.3) contains several parameters, and all asymptotic methods of constructing an approximate solution are based on certain a priori assumptions concerning the form of the functional dependence of the solution on the parameters. Usually the solution is sought in the form of a formal series in powers of a small parameter whose choice is dictated by physical considerations. The formal series will be asymptotic for the exact solution only when the small parameter is correctly chosen.

In a number of papers (/5–7/ et al) the Korteweg-de Vries method was used to deal with the problems of waves formed when a viscous film flowed down an inclined plane. The Korteweg-de Vries type equations were used in /7–9/ to study the stability of a one-dimensional flow and of the steady, two-dimensional, periodic and soliton-type solutions. Other references of similar type can be found in /5–9/. In /10/ a formal expansion in terms of a small parameter was applied to the stationary system (2.1)–(2.3) under the assumption that the Reynolds number was small and $\alpha\approx\pi/2$, but correct formulas for the wave profile and the dependence of the rate of propagation on the amplitude were not obtained.

Below we use the method of /11, 12/, which is based on reducing the problem (2.1)–(2.3) to a boundary value problem for a region with a known boundary.

By virtue of the conditions (2.3) written in terms of the independent variables x, ψ a rectilinear strip corresponds to the region of flows. It is convenient to replace the independent variable ψ by the independent variable η connected with ψ by the relation (1.2). Then the strip $-\infty<x<+\infty$, $0\leqslant\eta\leqslant 1$ will correspond to the region of flow, while the ordinate $y(x, \eta)$ will become the dependent variable. Differentiating the identity $y(x, \eta(\psi(x, y)))\equiv y$, we obtain the relations

$$\frac{\partial\psi}{\partial y}=\frac{a(\eta)}{y_\eta(x,\eta)},\quad \frac{\partial\psi}{\partial x}=-\frac{a(\eta)\,y_x(x,\eta)}{y_\eta(x,\eta)} \tag{2.4}$$

$$z = \Delta\psi = \frac{1}{y_\eta}\frac{\partial}{\partial\eta}\left(\frac{a}{y_\eta}\right) - \frac{\partial}{\partial x}\left(\frac{ay_x}{y_\eta}\right) + \frac{y_x}{y_\eta}\frac{\partial}{\partial\eta}\left(\frac{ay_x}{y_\eta}\right)$$

Since by virtue of $(1.1)\, a(1) = c$, we obtain from (2.3) the boundary conditions for $y(x, \eta)$

$$y(x, 1) = y_\eta(x, 1) = 1 \tag{2.5}$$

If we now take the function $\omega(x, \eta)$, expressed in terms of $y(x, \eta)$ as follows:

$$y(x, \eta) = \eta - \frac{\omega(x, \eta)}{a(\eta)} - \int_1^\eta \varphi\left[\left(\frac{\omega(x, t)}{a(t)}\right)_t\right] dt \tag{2.6}$$

$$\omega(x, 1) = \omega_\eta(x, 1) = 0, \quad \varphi(u) = e^{-u} - 1 + u$$

as the dependent variable, the boundary conditions (2.5) will be satisfied.

We can now reduce the system of Navier-Stokes equations (2.1) and boundary conditions (2.2) to the solution of an integrodifferential equation with the initial condition when $\eta = 0$

$$\frac{\partial^3\omega}{\partial\eta^3} = 3\varphi\left[\left(\frac{3\omega}{a}\right)_\eta\right] - 2a\left(\frac{\omega}{a}\right)^2_{\eta\eta} + F(y)\exp\left[-\left(\frac{3\omega}{a}\right)_\eta\right] \tag{2.7}$$

$$\eta = 0, \quad \omega_{\eta\eta} - \mu\omega = a\chi(y)\exp(-2u_\eta), \quad \mu = 2/\lambda$$

$$F(y) = \left\{-3\operatorname{ctg}\alpha\, y_x + 2a\frac{\partial}{\partial x}\left[\frac{1}{y_\eta}\frac{1 + y_x^2}{1 - y_x^2}\frac{\partial}{\partial\eta}\left(\frac{y_x}{y_\eta}\right)\right]\right\}_{\eta=0} +$$

$$\int_0^\eta\left[\frac{\partial}{\partial x}(z_\eta y_x - z_x y_\eta) - Ra^2\frac{\partial^2}{\partial x^2}\left(\frac{y_x}{y_\eta}\right)\right] d\eta + \frac{Ra^2}{2}\frac{\partial}{\partial x}\left(\frac{y_x}{y_\eta}\right)^2 -$$

$$R\frac{a^2}{y_\eta^3}y_{x\eta} - \frac{z_\eta}{y_\eta}y_x^2 - \frac{1}{y_\eta}\frac{\partial}{\partial\eta}\left[z - \frac{1}{y_\eta}\frac{\partial}{\partial\eta}\left(\frac{a}{y_\eta}\right)\right] - z_x y_x$$

$$\chi(y) = -\frac{\partial}{\partial x}\left(\frac{y_x}{y_\eta}\right) + \frac{1}{2}\frac{\partial}{\partial\eta}\left(\frac{y_x}{y_\eta}\right)^2 - \frac{2}{1 - y_x^2}\frac{\partial}{\partial x}\left(\frac{y_x}{y_\eta}\right)^2$$

Relation (2.6) gives the expression for y in terms of ω. The boundary conditions for $\omega(x, \eta)$ are given in (2.6), and $z(y)$ is given by (2.4).

We note that the equations and boundary conditions are invariant with respect to the displacement in the indepent variable x; therefore the solution is also obtained apart from the displacement. To remove this indeterminancy, we shall require the y axis should pass through the maximum and minimum of the profile of the free boundary. This leads to the following condition:

$$y_x(0, 0) = 0 \tag{2.8}$$

We shall also separate the linear parts of the operators $F(y)$ and $\chi(y)$

$$F_{\text{lin}}(y) = 2\operatorname{ctg}\alpha\frac{\partial\omega(x, 0)}{\partial x} - \frac{\partial^3\omega(x, 0)}{\partial x^2\,\partial\eta} - 2\frac{\partial^3\omega(x, \eta)}{\partial x^2\,\partial\eta} + \tag{2.9}$$

$$\int_0^\eta Ra\frac{\partial^3\omega}{\partial x^3}d\eta - \int_0^\eta\frac{\partial^4\omega}{\partial x^4}d\eta + R\frac{\partial}{\partial x}\left(a\frac{\partial\omega}{\partial\eta} - a'\omega\right), \quad a\chi_{\text{lin}} = \frac{\partial^2\omega}{\partial x^2}$$

3. Constructing an approximate solution. The form of the equation and boundary conditions (2.7) and (2.6) enables us to apply the general scheme of constructing the long-wave theory /1—3/. Neglecting in the equations and bounary conditions the non-linear terms and terms containing derivatives in x, we arrive at the following eigenvalue problem:

$$v'''(\eta) = 0, \quad v''(0) - \mu_0 v(0) = 0, \quad v(1) = v'(1) = 0 \tag{3.1}$$

It can be confirmed that problem (3.1) has a unique eigenvalue and eigenfunction

$$\mu_0 = 2, \quad v_0(\eta) = \frac{3}{2}(\eta - 1)^2 = a_0 - 3\eta, \quad a_0 = \frac{3}{2}(\eta^2 + 1) \tag{3.2}$$

From the relation $\mu = 2/\lambda$ it follows that

$$\lambda_0 = \frac{1}{2}\mu_0 = 1, \quad c_0 = \frac{3}{2}(1 + \lambda_0) = 3$$

Thus the critical wave propagation velocity is equal to three, which agrees with the result obtained in /10/.

We shall assume that the angle of inclination of the channel bottom satisfies the condition $\alpha \geqslant \alpha_0 > 0$. Let us write $\mu = 2 - \varepsilon$, $R = R_0 + \varepsilon R_1$.

The small parameter ε characterizes the nearness of the velocity of propagation to its critical value, since

$$c = \frac{3}{2}(1+\lambda) = 3 + \frac{3\varepsilon}{4-2\varepsilon}$$

The critical value of the Reynolds number R_0 will be found below.

Following the long-wave theory we shall expand the independent variable x and seek the solution in the form of a series in powers of the parameter $\sqrt{|\varepsilon|}$

$$x = \xi\,|\varepsilon|^{1/2}, \quad \omega = |\varepsilon|\,\omega_1 + |\varepsilon|^{3/2}\omega_2 + |\varepsilon|^2\omega_3 + \ldots \tag{3.3}$$

Substituting the expansion (3.3) into the equations and boundary conditions (2.6), (2.7), we obtain a sequence of boundary value problems for determining the functions $\omega_i(\xi, \eta)$. For ω_1 we obtain the following boundary value problem:

$$\frac{\partial^3\omega_1}{\partial\eta^3} = 0, \quad \frac{\partial^2\omega_1}{\partial\eta^2}(\xi, 0) - 2\omega_1(\xi, 0) = 0, \quad \omega_1(\xi, 1) = \frac{\partial\omega_1}{\partial\eta}(\xi, 1) = 0.$$

By virtue of (3.1) and (3.2) its solution has the form

$$\omega_1(\xi, \eta) = C(\xi)\, v_0(\eta) \tag{3.4}$$

where $C(\xi)$ is an unknown function which will have to be determined from the subsequent approximations.

We obtain the following boundary value problem for determining ω_2:

$$\frac{\partial^3\omega_2}{\partial\eta^3} = R_0\left(\frac{\partial\omega_1}{\partial\eta}a_0 - a_0'\omega_1\right)_\xi + 2\,\mathrm{ctg}\,\alpha\,\frac{\partial\omega_1(\xi, 0)}{\partial\xi} = \tag{3.5}$$

$$C'(\xi)E(\eta), \quad E(\eta) = 3\,\mathrm{ctg}\,\alpha - \frac{9}{2}R_0(1-\eta^2)$$

$$\frac{\partial^2\omega_2}{\partial\eta^2}(\xi, 0) - 2\omega_2(\xi, 0) = 0, \quad \omega_2(\xi, 1) = \frac{\partial\omega_2}{\partial\eta}(\xi, 1) = 0$$

Simple calculations prove the validity of the following lemma.

Lemma. Let $f(\eta)$ be a function continuous in the interval $[0, 1]$, and let α be a real number. The inhomogeneous boundary value problem

$$d^3v/d\eta^3 = f(\eta), \quad v''(0) - 2v(0) = \alpha, \quad v(1) = v'(1) = 0$$

is solvable if and only if the condition

$$\alpha + \int_0^1 (1-t^2)\, f(t)\, dt = 0 \tag{3.6}$$

holds.

Applying the lemma to the boundary value problem (3.5), we obtain the condition of its solvability in the form

$$\int_0^1 (1-t^2)\, E(t)\, dt = 0$$

Substituting the expression for the function $E(t)$ and (3.5) we find that the critical value of the Reynolds number is $R_0 = {}^5/_6\,\mathrm{ctg}\,\alpha$, and the one-dimensional flow loses its stability at this value (see e.g. /8/). If we take the maximum velocity of the unperturbed flow as the unit velocity instead of the mean velocity, then $R_0 = {}^5/_4\,\mathrm{ctg}\,\alpha$.

We obtain the following expression for the function ω_2:

$$\omega_2(\xi, \eta) = C'(\xi)\,\delta(\eta) + D(\xi)\, v_0(\eta), \quad \delta(\eta) = \frac{3}{48}(2\eta^3 - \eta^5 - \eta) \tag{3.7}$$

where $D(\xi)$ is a new unknown function. Thus the second approximation equations yield the critical value of the Reynolds number, but not the unknown function $C(\xi)$.

Let us now formulate the boundary value problem for determining the function $\omega_3(\xi, \eta)$. Using expressions (3.4) and (3.7), we obtain the following formulas for the functions $\omega_1(\xi, \eta)$ and $\omega_2(\xi, \eta)$:

$$\frac{\partial^3\omega_3}{\partial\eta^3} = A(\eta)\,C(\xi)^2 + B(\eta)\,C''(\xi) + E(\eta)\,D'(\xi) \tag{3.8}$$

$$\omega_3(\xi, 1) = \frac{\partial\omega_3(\xi, 1)}{\partial\eta} = 0$$

$$\frac{\partial^2\omega_3(\xi, 0)}{\partial\eta^2} - 2\omega_3(\xi, 0) = \frac{3}{2}\,\mathrm{sgn}\,\varepsilon\, C(\xi) + \frac{3}{2}C''(\xi)$$

$$A(\eta) = 54\frac{(1-\eta^2)^2}{(1+\eta^2)^4} - 48\frac{\eta^2(3-\eta^2)^2}{(1+\eta^2)^5}$$

$$B(\eta)=9-6\eta+\frac{5}{64}\operatorname{ctg}^2\alpha(-1+7\eta^2-3\eta^4-3\eta^6)$$

Applying the condition of solvability (3.6) to problem (3.8), we obtain a differential equation for determining the unknown function $C(\xi)$. We must remember here that the function $E(\eta)$ is orthogonal to the function $(1-\eta^2)$

$$a_1C''(\xi)+a_2C'(\xi)+a_3C(\xi)=0 \tag{3.9}$$

$$a_1=\frac{3}{2}+\int_0^1 B(\eta)(1-\eta^2)\,d\eta=9,\quad a_2=\int_0^1 A(\eta)(1-\eta^2)\,d\eta=6,$$

$$a_3=\frac{3}{2}\operatorname{sgn}\varepsilon$$

The second integral in (3.9) is obtained by substitution $\eta=\operatorname{tg}\varphi/2$.

Let us turn our attention to the fact that the coefficients of (3.9) are independent of the angle of inclination α of the channel bottom. Equation (3.9) has the form $C''(\xi)+{}^2/_3C^2(\xi)={}^1/_6\operatorname{sgn}\varepsilon C(\xi)$ and has been studied more than once (see e.g. /1–3/). Periodic solutions exist only when $\varepsilon>0$, and can be expressed in terms of the elliptic Jacobi function $\operatorname{cn}(\xi,k)$. When $k\to 1$, and hence when the period tends to infinity, the periodic solution degenerates into a periodic (a soliton). In this case we have

$$C(\xi)=\operatorname{sech}^2\left(\frac{\xi}{2\sqrt{6}}\right) \tag{3.10}$$

The formulas for the free boundary and velocity propagation have the form

$$Y(x)=-\frac{3\varepsilon}{8}\operatorname{sech}^2\left(\frac{x}{2}\sqrt{\frac{\varepsilon}{6}}\right)+o(\varepsilon),\quad c=3+\frac{3\varepsilon}{4}+o(\varepsilon) \tag{3.11}$$

and in dimensional variables

$$Y(x)=-a\operatorname{sech}^2\left(\frac{x}{3}\sqrt{\frac{a}{H^3}}\right),\quad c=\frac{Q}{H}\left(3+\frac{2a}{H}\right),\ a>0 \tag{3.12}$$

Since the solutions of (3.9) are obtained apart from the displacement, we choose the solution satisfying the condition (2.8).

We find that to a first approximation the wave profile and its velocity propagation depend only on the dimensionless amplitude and are independent of the angle of inclination of the channel bottom and the Reynolds number. This is explained by the relation connecting the parameters: $FR\sin\alpha=3$, $R={}^5/_6\operatorname{ctg}\alpha+O(\varepsilon)$, $c=3+O(\varepsilon)$.

The first formula of (3.12) can be obtained from the expression for the wave profile given in /5/.

4. Constructing the subsequent approximations. Solving the boundary value problem (3.8) we obtain

$$\omega_3(\xi,\eta)=\varphi(\xi,\eta)+D'(\xi)\,\delta(\eta)+K(\xi)\,v_0(\eta)$$

where the function $\varphi(\xi,\eta)$ is known, the function $\delta(\eta)$ is given by Eq.(3.7), and $K(\xi)$ is a new unknown function. The boundary value problem for determining ω_4 has the form

$$\begin{gathered}\frac{\partial^3\omega_4}{\partial\eta^3}=27\left(\frac{\omega_1}{a_0}\right)_\eta\left(\frac{\omega_2}{a_0}\right)_\eta-4a_0\left(\frac{\omega_1}{a_0}\right)_{\eta\eta}\left(\frac{\omega_2}{a_0}\right)_{\eta\eta}-2\frac{\partial^3\omega_2}{\partial\xi^2\partial\eta}+\\ R\frac{\partial}{\partial\xi}\left(a_0\frac{\partial\omega_3}{\partial\eta}-a_0'\omega_3\right)+2\operatorname{ctg}\alpha\,\frac{\partial\omega_3(\xi,0)}{\partial\xi}-\\ \frac{\partial^3\omega_2(\xi,0)}{\partial\xi^2\partial\eta}+\varphi_1(\xi,\eta),\quad \omega_4(\xi,1)=\frac{\partial\omega_4(\xi,1)}{\partial\eta}=0\\ \frac{\partial^2\omega_4(\xi,0)}{\partial\eta^2}-2\omega_4(\xi,0)=-\operatorname{sgn}\varepsilon\omega_2(\xi,0)+\frac{\partial^2\omega_2(\xi,0)}{\partial\xi^2}+\varphi_2(\xi)\end{gathered}$$

where $\varphi_1(\xi,\eta)$ and $\varphi_2(\xi)$ are known functions. It can easily be shown that these functions will be polynomials in $C(\xi)$. Using the condition of solvability of (3.6) we obtain the equation for determining the unknown function $D(\xi)$

$$D''(\xi)+\frac{4}{3}C(\xi)D(\xi)-\frac{1}{6}\operatorname{sgn}\varepsilon D(\xi)=\psi(\xi) \tag{4.1}$$

where $\psi(\xi)$ is a known function (a polynomial in $C(\xi)$).

Equations of type (4.1) were studied in /1–3/. The relation $y_1 = C'(\xi)$, represents one particular solution of the homogeneous equation, and the other (y_2) can be found with help of Liouville's formula. If $C(\zeta)$ is a periodic solution of (3.9), we have $y_2(\xi) = \zeta(\xi) + B\xi C'(\xi)$, where $\zeta(\xi)$ is an even periodic function and B is a constant depending on the period. The periodic solution of (4.1) satisfying the condition $D'(0) = 0$, has the form

$$D(\xi) = y_2(\xi)\int_L^{\xi}\psi(t)\,y_1(t)\,dt - y_1(\xi)\int_0^{\xi}\psi(t)\,y_2(t)\,dt + \frac{y_2(\xi)}{2BL}\int_{-L}^{L}\psi(t)\,y_2(t)\,dt$$

Note that in order to construct further approximations, we shall have to solve Eq.(4.1) every time, but with different right-hand sides.

Thus we can construct, one after the other, all terms of the series (3.3) which, generally speaking, will not converge. By analogy with the long-wave theory in a perfect fluid, we can expect that it will be uniformly asymptotic for the exact solution as $\varepsilon \to 0$.

We also note that if the Reynolds number is small and the angle of inclination α is nearly $\pi/2$, or more accurately $\operatorname{ctg}\alpha = o(\varepsilon)$, $R = o(\varepsilon)$ as $\varepsilon \to 0$, then the solution of the problem must be sought in the form of a series in integer powers of the parameter ε, $\omega = \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots$. To a first approximation we have $\omega_1(\xi, \eta) = C(\xi)\,v_0(\eta)$, where the unknown function $C(\xi)$ is found from the second approximation equations. The equations for determining $\omega_2(\xi, \eta)$ have the form (3.8), provided that we write in these equations $\operatorname{ctg}\alpha = 0$. The equation for determining the free boundary will retain the form (3.12), but in this case it will no longer be necessary to satisfy the relation $R = {}^5/_6 \operatorname{ctg}\alpha$ connecting the Reynolds number with the angle of inclination. For small angles of inclination and large Reynolds numbers another asymptotic theory will have to be developed, related to bounday layer theory, but this problem will not bh considered here (see e.g. /13/.

5. Some thoughts on the proof of the theorems of existence and uniqueness. A general scheme for proving the theorems of existence and uniqueness of long waves degenerating into solitons as the wavelength tends to infinity, was developed in /1–3/. The scheme can be used, after some modifications, in the theory of oscillating waves, but the technical complications become greater.

Let us retain in (2.7) only the linear and quadratic terms not containing the derivatives in x. We shall assume for simplicity that the Reynolds number is small (therefore the angle α will be nearly $\pi/2$). Thus we neglect in (2.7) terms containing R and $\operatorname{ctg}\alpha$. Taking into account (2.9) we obtain

$$\partial^3\omega/\partial\eta^3 = L_0\omega + F_{\text{lin}}\omega, \quad \omega(x, 1) = \omega_\eta(x, 1) = 0 \tag{5.1}$$

$$F_{\text{lin}} = -\frac{\partial^3\omega(x, 0)}{\partial x^2\,\partial\eta} - 2\frac{\partial^3\omega(x, \eta)}{\partial x^2\,\partial\eta} - \int_0^{\eta}\frac{\partial^4\omega}{\partial x^4}\,d\eta$$

$$L_0\omega = \frac{27}{2}\left(\frac{\omega}{a_0}\right)_{\eta}^{2} - 2a_0\left(\frac{\omega}{a_0}\right)_{\eta\eta}^{2}$$

$$\omega_{\eta\eta}(x, 0) - 2\omega(x, 0) = -\varepsilon\omega(x, 0) + \frac{\partial^2\omega(x, 0)}{\partial x^2} \tag{5.2}$$

The problem of solving the integrodifferential equation (5.1) with boundary conditions (5.2) and (2.6), can be replaced by the equivalent boundary value problem for a fourth-order differential equation, Differentiating (5.1) with respect to η, we obtain

$$\Delta^2\omega = \frac{\partial}{\partial\eta}(L_0\omega) \tag{5.3}$$

where Δ is the Laplace operator, Substituting into (5.1) $\eta = 0$, we obtain the boundary conditions

$$\frac{\partial^3\omega}{\partial\eta^3} = -3\frac{\partial^3\omega}{\partial x^2\,\partial\eta} + L_0\omega, \quad \eta = 0 \tag{5.4}$$

Thus we have to solve the fourth-order equation (5.3) with boundary conditions (5.2), (5.4) and (2.6).

Similar problems for some classes of elliptic second-order equations were studied in /1–3/ using the method of splitting based on projecting the function ω onto the direction of the eigenfunction $v_0(\eta)$ and its orthogonal complement. Since equation (5.3) contains mixed derivatives and the corresponding non-selfconjugate differential operators, it follows that the method cannot be applied directly to equation (5.3).

We can follow the more complex method used in /11, 12/. We shall consider, to be specific, the case of a solitary wave. Let us consider a linear, inhomogeneous boundary value problem for a biharmonic operator in the strip $-\infty < x < +\infty,\ 0 \leqslant \eta \leqslant 1$

$$\Delta^2 u = \frac{\partial f(x,\eta)}{\partial\eta}, \quad u(x,1) = \frac{\partial u}{\partial\eta}(x,1) = 0 \tag{5.5}$$
$$\frac{\partial^3 u(x,0)}{\partial\eta^3} + 3\frac{\partial^3 u(x,0)}{\partial x^2\partial\eta} = f(x,0)$$
$$\frac{\partial^2 u(x,0)}{\partial\eta^2} - \frac{\partial^2 u(x,0)}{\partial x^2} - 2u(x,0) = \varphi(x)$$

where $\varphi(x)$ and $f(x,\eta)$ are fairly smooth functions decreasing exponentially as $x \to \infty$.

If we construct Green's function for problem (5.5), we can reduce the non-linear (5.3) to a non-linear integrodifferential equation. Applying a Fourier transformation, we can express Green's function of problem (5.5) in terms of a contour integral of a meromorphic function with a multiple pole at the zero. Using the theory of residues, we can represent Greens' function in the form of the sum of a certain series all of whose terms (expect the first corresponding to the residue at the zero) decrease exponentially at infinity. The splitting of the integrodifferential equation will correspond to the splitting of Green's function. Further proof can be carried out using the scheme in /1—3/. The technical complications are considerable, but can be overcome.

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